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Fair Indirect Majority Rules

by

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Consider a situation in which n members of a group are asked to determine whether a proposition is true or false. In the simplest case, where all make their decisions independently of each other, a straightforward majority rule is best in the sense of maximizing the probability of a correct group decision. Where, however, there is a substantial degree of statistical dependence among the group members' decisions, other rules may

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be better.

A model of individual decision making is considered, assuming a possibly strong correlation among members of certain subgroups. It is shown that some indirect majority rules (e.g. the electoral college system) and intermediate rules may in such case be better than direct majority rule.

"FAIR" INDIRECT MAJORITY RULES

Guillermo Owen

Consider a situation in which n members of a group are asked to determine whether a proposition is true or false. We shall assume, for Bayesian purposes, that the *a priori* probability of truth or falsehood is 1/2. Moreover we assume that each individual has a probability $p > 1/2$ of giving a correct answer (and that this p is the same for all). The question is whether there would be any reason to use other than a straightforward majority rule for this purpose.

As an example, consider the case $n = 5$. Straightforward majority rule means that the group will accept as correct the opinion of any three of them.

Suppose, however, that three of the members show a strong positive correlation in their judgments; the other two decide independently. In that case, it is not too difficult to see that, for sufficiently high correlation among the first three, a positive 3-2 vote (pitting the first three against the other two) should not give rise to a positive group judgment; in fact the two negative votes represent two negative bits of information, whereas the three positive votes might represent as little as one positive bit.

The above decision rule is, of course, not "fair": the three first decision-makers are treated worse than the other two. Unless there is reason to expect this type of behavior on the part of the group members, this decision rule might be open to serious (e.g., constitutional) objections.

There are, however, "fair" group decision rules which are not direct majority rules. ("Fair" means, in mathematical terms, that the winning coalitions allow a transitive group of permutations over the set of all individual voters.) The best known such rule would be the <u>indirect majority decision rule</u>: the group of n members is divided into m subgroups with n/m members each; each subgroup decides by majority vote among its n/m members, and the entire group decides by a majority of the m subgroups.

Whether such a rule is more efficient (i.e., is more probably correct) than straightforward majority rule will depend on the voting correlations among the several individuals. Let us therefore consider the following model. Each of the m subgroups has a "Group spirit" (possibly a group leader) which makes a decision; these m decisions each have probability p of being correct, and they are independent. Now each of the k members of the j^{th} subgroup will either (1) with probability α_j, follow his leader, or (2) with probablity $1 - \alpha_j$, make an independent decision, which also has probability p of being correct. According to this model, two members of the same (j^{th}) subgroup make correlated decisions, with correlation α_j^2; members of two different subgroups act independently. Each member has probability p of being right. The number

α_j $(0 \leq \alpha_j \leq 1)$ represents, somehow, the "charisma" of the group leader, or the cohesiveness of the group.

In this model, it might seem that, for sufficiently high values of the α_j, the indirect majority rule could be better than the direct majority rule. In fact, however, for equal values of α_j this is usually not so. For, as an example, suppose we have a group with three subgroups (of equal size). Assume subgroup 1 votes 80% positive, while subgroups 2 and 3 both vote 60% negative. In this case, the groups divide 2-1 negative, but the individual vote is 53% positive. It turns out that the "most likely" interpretation is that the common value of the α_j's is near 0.4; the group leaders have voted 2-1 negatively, but the independently voting members are voting positively, 2-1. Thus the weight of the evidence is positive.

Let us suppose, however, that the α_j are not all equal. "Fairness" requires that the several subgroups be treated equally, but we can do this by assuming that the α_j are drawn independently from an identical distribution--perhaps a uniform distribution over the unit interval. (The interpretation is that each group has a leader, but some leaders turn out to be more charismatic than others.) In this interpretation, the heavy positive voting in subgroup 1 seems to mean that its leader is very popular, i.e., α_1 is very high while α_2 and α_3 are low.

Let us see how this works. There are essentially two possibilities: the proposition is either true or false.

(1) The proposition is true. Then the group leader for subgroup 1 is right and the other two are wrong. The probability of this is pq^2. Assuming n is very large, the fraction s_1 of members of group 1 voting positively is

$$s_1 = \alpha_1 + p(1-\alpha_1) = \alpha_1 q + p$$

(where $q = 1-p$) while the fractions s_2 and s_3 voting positively will be

$$s_j = (1-\alpha_j)p \qquad (j = 2,3).$$

This gives us densities $1/q$ for s_1, and $1/p$ for s_2 and s_3, all independent. The joint density of (s_1, s_2, s_3) then is $pq^2/qp^2 = q/p < 1$.

(2) The proposition is false. Similar reasoning tells us that the joint density is $p/q > 1$.

Assuming a uniform Bayesian prior, then, the posterior distribution assigns probability $\dfrac{p^2}{p^2 + q^2}$ to case (2): the proposition is false. This is greater than $1/2$ (since $p > q$); thus the indirect majority rule gives a better result than the direct majority rule.

More generally, suppose we have a division of a group into subgroups. Instead of an "indirect majority rule" such as described above, we might consider as rule which assigns a certain number of points, either positive or negative, to the result within each group, depending on the number of positive and negative votes within the group. In other words, we would have

$$w_j = g_j(n_j, k_j)$$

where n_j is the number of indidviduals in subgroup j, and k_j the number of these voting affirmatively in that subgroup. Then the whole group will accept the proposition if

$$W = \sum_{j=1}^{m} w_j$$

is positive, and reject if $W < 0$.

The simpliest case, of course, is given by $w_j = 2k_j - n_j$: it corresponds to direct majority rule. The indirect majority rule is given by

$$g_j(n_j,k_j) = \begin{cases} 1 & \text{if} \quad 2k_j > n_j \\ -1 & \quad\quad 2k_j < n_j \\ 0 & \quad\quad 2k_j = n_j . \end{cases}$$

Other rules may however be used: we would expect simply that g_j be monotone as a function of k_j for each n_j, and (for symmetry) that

$$g_j(n_j,k_j) = -g_j(n_j,n_j-k_j).$$

We might also ask that g_j be concave in k_j for $k_j > \frac{n_j}{2}$, although that is not strictly necessary.

It remains to see how the rules g_j should be chosen.

To see how this can be done, let us assume that, in a given subgroup each individual has probability p of a correct vote, and that the voting correlations among the several individuals are maintained constant. In

that case, a given configuration, A, of correct votes has a probability $H_A(p)$, and the complementary configuration has probability $H_A(1-p)$ of occurring. The conditional probability that a positive vote is correct, given that the configuration A voted positive, is then

$$P(\text{pos} \mid A) = \frac{H_A(p)}{H_A(p) + H_A(1-p)}$$

and clearly, the probability that a negative vote is correct woud then be

$$P(\text{neg} \mid A) = \frac{H_A(1-p)}{H_A(p) + H_A(1-p)}$$

The weight to be given to such a configuration is then

$$w(A) = \log \frac{P(\text{pos} \mid A)}{P(\text{neg} \mid A)}$$

or consequently

$$cw(A) = \log H_A(p) - \log H_A(1-p) ,$$

where c is a constant of proportionality.

Suppose, now, that p is slightly greater than 1/2: $p = 1/2 + r$, where r is small. These are generally the interesting cases, as they represent situations where opinion will likely be closely divided. We have then

$$cw(A) = \log H_A(\tfrac{1}{2} + r) - \log H_A(\tfrac{1}{2} - r)$$

or

$$\frac{cw(A)}{r} \cong \frac{2H_A'(\frac{1}{2})}{H_A(\frac{1}{2})}$$

so that $w(A)$ should be chosen proportional to $H_A'(\frac{1}{2})/H_A(\frac{1}{2})$. To determine the constant of proportionality we note that, if the subgroup reduces to a single individual, there is only one possible configuration, and we will have

$$H(p) = p : H'(p) = 1$$

so that

$$\frac{H'(\frac{1}{2})}{H(\frac{1}{2})} = \frac{1}{\frac{1}{2}} = 2 .$$

Hence, if we assign weight 1 to a single positive vote, we should assign weight

$$w(A) = \frac{1}{2} \frac{H_A'(\frac{1}{2})}{H_A(\frac{1}{2})}$$

to a positive vote by the subset A, with negative votes by the complementary subset.

Suppose, now, that a group with N individuals is divided into m subgroups, with $n_1, n_2, \ldots, n_m$ individuals respectively. We assume once again that all individuals have equal probability p of voting correctly.

Suppose that in a given subgroup, with n members, k of them vote affirmatively, while the remaining $n-k$ vote negatively. Assuming an index α of correlation (as discussed above), the probability that exactly the given k vote correctly is

$$h(p;\alpha) = p(\alpha+\beta p)^k\ (\beta q)^{n-k} + q(\beta p)^k\ (\alpha+\beta q)^{n-k}$$

where $q = 1-p$, $\beta = 1-\alpha$. This can be rewritten as

$$h(p;\alpha) = pq^{n-k} (p + q\alpha)^k (1-\alpha)^{n-k} + qp^k (1-\alpha)^k (q + p\alpha)^{n-k}.$$

If α is known this can be evaluated directly, as can its derivative. Suppose, however, that α is assumed distributed according to the density function $f(\alpha)$, $0 \leq \alpha < 1$. Then we would set

$$H(p) = \int_0^1 h(p;\alpha)\, f(\alpha)\, d\alpha$$

and this may or may not be easy to evaluate, depending on the form of f.

We will consider here the one-parameter family of distributions

$$f(\alpha) = (v+1)(1-\alpha)^v \qquad 0 \leq \alpha \leq 1$$

where $v > -1$ is a parameter. We would then have

$$H(p) = (v+1)\, pq^{n-k} \int_0^1 (p+q\alpha)^k (1-\alpha)^{n-k+v}\, d\alpha$$

$$+ (v+1)\, qp^k \int_0^1 (1-\alpha)^{k+v} (q+p\alpha)^{n-k}\, d\alpha .$$

The change of variables $u = p + q\alpha$ transforms this to

$$H(p) = (v+1)\, pq^{-v-1} \int_p^1 u^k (1-u)^{n-k+v}\, du$$

$$+ (v+1)\, qp^{-v-1} \int_q^1 (1-u)^{k+v} u^{n-k}\, du .$$

Some calculus will now give us

$$H'(p) = \frac{(v+1)(vp+1)}{q^{v+2}} \int_p^1 u^k (1-u)^{n-k+v}\, du$$

$$- \frac{(v+1)(vp+1)}{q^{v+2}} \int_q^1 (1-u)^{k+v} u^{n-k}\, du$$

$$- p^{k+1} q^{n-k-1} + p^{k-1} q^{n-k+1} .$$

Evaluating at $p = 1/2$, we will have

$$H(\tfrac{1}{2}) = (v+1)2^v \int_{\frac{1}{2}}^1 [u^k(1-u)^{n-k+v} + (1-u)^{k+v} u^{n-k}]\, du$$

and

$$H'(\tfrac{1}{2}) = (v+1)(v+2)2^{v+1} \int_{\frac{1}{2}}^1 [u^k(1-u)^{n-k+v} - (1-u)^{k+v} u^{n-k}]\, du$$

The desired weighting factor would then be

$$w = \frac{H'(\frac{1}{2})}{2H(\frac{1}{2})}$$

or

$$w = (v+2) \frac{\int_{1/2}^1 u^k(1-u)^{n-k+v}\, dv - \int_{1/2}^1 (1-u)^{k+v} u^{n-k}\, du}{\int_{1/2}^1 u^k(1-u)^{n-k+v}\, dv + \int_{1/2}^1 (1-u)^{k+v} u^{n-k}\, du}$$

Students of probability will of course recognize the integrals in this last expression as the incomplete beta function, defined as

$$B_q(a,b) = \int_0^q u^{a-1} (1-u)^{b-1}\, du$$

our expression is then

$$w = (v+2)\,\frac{B_{1/2}(n-k+v+1,k+1) - B_{1/2}(k+v+1,n-k+1)}{B_{1/2}(n-k+v+1,k+1) + B_{1/2}(k+v+1,n-k+1)}$$

Unfortunately, the incomplete beta function is not extensively tabulated. For small values of n, it is of course possible to evaluate the integrals directly (analytically) as the integrands reduce to polynomials of degree n. For large values of n, the integrals must be evaluated numerically, though certain approximations, related to the normal distribution function are also available [Abramowitz & Segun, 1964].

It is not too difficult to see that, if k is considerably larger than $n/2$, then w will approach $v+2$ asymptotically; clearly $w \leq v+2$ at all times. We note that the parameter v can be taken as, in some sense, a measure of the subgroup's a priori independence: large values of v mean that α will tend to be small, and vice-versa. Thus if v is large, a large margin for one or the other possibilities is given relatively heavy weight--coming as it does from individuals who tend to choose independently.

As an example, consider the case of a group divided into subgroups of 15 individuals each. The Table 1, shows values of the functions

$$w_j = g(15,k\ ;\ v)$$

for $8 \leq k \leq 15$ and $v = 0,1,2$. As may be seen, for $v = 2$, an 8-7 margin should be weighed 0.6, while a 9-6 margin is weighed 1.7 --not quite three times as much as an 8-7 edge. As k increases to 15, the weight approaches, but never quite reaches, the limiting value $v+2 = 4$. Similar considerations may be seen throughout the table.

TABLE 1

n = 15

k	v = 0	v = 1	v = 2
8	0.393	0.514	0.604
9	1.091	1.457	1.738
10	1.580	2.183	2.672
11	1.846	2.642	3.332
12	1.957	2.875	3.722
13	1.992	2.967	3.908
14	1.999	2.994	3.977
15	2.000	2.999	3.996

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